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TOUCHING RANDOM SURFACES AND LIOUVILLE GRAVITY

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ABSTRACT

Large N matrix models modified by terms of the form $g(\text{Tr } \Phi^n)^2$ generate random surfaces which touch at isolated points. Matrix model results indicate that, as g is increased to a special value g_t , the string susceptibility exponent suddenly jumps from its conventional value γ to $\frac{\gamma}{\gamma-1}$. We study this effect in Liouville gravity and attribute it to a change of the interaction term from $Oe^{\alpha_+\phi}$ for $g < g_t$ to $Oe^{\alpha_-\phi}$ for $g = g_t$ (α_+ and α_- are the two roots of the conformal invariance condition for the Liouville dressing of a matter operator O). Thus, the new critical behavior is explained by the unconventional branch of Liouville dressing in the action.

1. Introduction

A remarkable aspect of the recent developments in two-dimensional quantum gravity has been an interplay between discretized [1] and continuum [2-5] approaches to the problem. The discretized approach, implemented mainly via elegant matrix model techniques, has so far proven more powerful [6]. However, the full significance of the matrix model results usually becomes apparent only after they are translated into the continuum language of Liouville gravity. By now there exists a considerable amount of evidence that the discretized and continuum approaches are indeed equivalent, but a general demonstration of this has not been found. We believe that much more can be learned from comparing the two approaches to two-dimensional quantum gravity.

While the matrix models which generate conventional discretized random surfaces have been investigated quite thoroughly, there exists a new class of matrix models where only some partial results are available. These models describe random surfaces which are allowed to touch each other at isolated points. This is implemented by adding a term of the form $g(\text{Tr } \Phi^n)^2$ to the action of an $N \times N$ hermitian matrix Φ .[★] The first matrix integral of this kind,

$$Z = \int \mathcal{D}\Phi e^{-N \left[\text{Tr} (\Phi^2 - \lambda \Phi^4) - \frac{g}{2N} (\text{Tr } \Phi^2)^2 \right]}, \quad (1.1)$$

was introduced in ref. [7]. The free energy,

$$F = \frac{\log Z}{N^2},$$

can be expanded in powers of $1/N^2$,

$$F = F_0 + F_1 N^{-2} + F_2 N^{-4} + \dots$$

Each term in this expansion has an interesting geometrical interpretation. Feynman graphs of the perturbation theory in λ generate the usual connected closed random surfaces, while the $g(\text{Tr } \Phi^2)^2$ term can glue a pair of such surfaces together at a point.

[★] n is often set equal to 2, but all finite n are expected to lead to the same universal behavior.

This point can be resolved into a tiny neck (a wormhole), so that the network of such touching surfaces can be assigned an overall genus. Thus, F_0 gives the sum over all such surfaces of overall genus zero (they look like trees of spherical bubbles such that any two bubbles touch at most once, and a bubble is not allowed to touch itself). In general, F_G is the sum over all surfaces of overall genus G .

The authors of ref. [7] found a critical line in the (λ, g) plane where the free energy becomes singular. For a fixed g , $F_0(\lambda)$ becomes singular at $\lambda = \lambda_c(g)$ on the critical line. There exists a critical value g_t such that, for $g < g_t$, the singularity is characterized by $\gamma = -1/2$, *i.e.*

$$F_0(\lambda) \sim (\lambda_c - \lambda)^{2-\gamma} \sim (\lambda_c - \lambda)^{5/2} .$$

In this phase the touching of random surfaces is irrelevant and one finds the conventional $c = 0$ behavior. For $g > g_t$, on the other hand, $\gamma = 1/2$, and one finds branched polymer behavior, which is dominated by the touching. Most interestingly, for $g = g_t$, the authors of ref. [7] found a new type of critical behavior with $\gamma = 1/3$. This is the first example of a matrix model where new critical behavior occurs due to fine-tuned touching interactions. We will generally refer to such new critical points as the modified matrix models.

The results above have been generalized to the k -th multicritical one-matrix model [8,9],

$$Z_k = \int \mathcal{D}\Phi e^{-N \left[\text{Tr } V_k(\Phi) - \frac{g}{N} (\text{Tr } \Phi^2)^2 \right]} , \quad (1.2)$$

where

$$V_k(\Phi) = \sum_{i=1}^k t_i \Phi^{2i} .$$

For $g = 0$, the parameters of the potential can be fine-tuned to give scaling behavior with $\gamma = -1/k$. As we increase g , then for some $g = g_t$ the scaling exponent suddenly jumps [8,9] to $\gamma_{\text{modified}} = 1/(k+1)$. These values of γ are puzzling because they are positive; in matrix models without the touching interactions only $\gamma \leq 0$ have been found.

Another piece of the puzzle is provided by the modified $c = 1$ matrix model, where a fine-tuning of g also leads to new critical behavior [10,11],

$$F_0 \sim \Delta^2 \log \Delta , \quad \Delta = \lambda_c - \lambda . \quad (1.3)$$

This should be contrasted with the conventional $c = 1$ scaling, $F_0 \sim \Delta^2 / \log \Delta$. The sum over spherical surfaces of fixed area A , obtained by an inverse Laplace transform of eq. (1.3), scales as $1/A^3$. In other words, the modified $c = 1$ matrix model has no scaling violations. An explanation of this effect in terms of Liouville gravity was proposed in ref. [11]. There it was argued that, while the conventional $c = 1$ scaling corresponds to the Liouville potential $\sim \phi e^{-\sqrt{2}\phi}$, at the new critical point the potential is $\sim e^{-\sqrt{2}\phi}$ instead.

In this paper we present a Liouville gravity explanation of all the new critical exponents obtained in the modified matrix models. This explanation is surprisingly simple and amounts to picking the unconventional branch in the gravitational dressing of the Liouville potential. For all the conventional matrix models describing (p, q) minimal models coupled to gravity, the correct scaling follows from the Liouville interaction of the form

$$\Delta \int d^2\sigma O_{\min} e^{\alpha_+ \phi} ,$$

$$\alpha_+ = \frac{1}{2\sqrt{3}}(\sqrt{1-c+24h_{\min}} - \sqrt{25-c})$$

where O_{\min} is the matter primary field of the lowest dimension,

$$h_{\min} = \frac{1 - (p - q)^2}{4pq} . \quad (1.4)$$

We will argue that the effect of fine-tuning the touching interaction is to replace the Liouville potential by

$$\Delta \int d^2\sigma O_{\min} e^{\alpha_- \phi} ,$$

$$\alpha_- = -\frac{1}{2\sqrt{3}}(\sqrt{1-c+24h_{\min}} + \sqrt{25-c}) .$$

It follows that the modified matrix models do not correspond to $c > 1$ string theories; they are simply new solutions of $c \leq 1$ string theories!

The structure of the paper is as follows. In section 2 we present the details of our Liouville gravity arguments. We reproduce the known matrix model results and make predictions for new calculations. In section 3 we confirm some of these predictions by finding the scaling behavior in the modified $(p, p+1)$ matrix models. In section 4 we discuss the directions for future work.

2. Fine-tuning in Liouville gravity

Let us consider (p, q) conformal minimal models coupled to quantum gravity. In the conformal gauge, the sum over surfaces of genus G is given by the path integral [5]

$$F_G = \int d\tau \int [d\Psi][d\phi][db][dc] e^{-S_\Psi - S_\phi - S_{b,c}} , \quad (2.1)$$

where S_Ψ is the matter action, $S_{b,c}$ is the standard ghost action, and τ collectively denotes the moduli. The action of the Liouville field is

$$S_\phi = \frac{1}{8\pi} \int d^2\sigma \left(\partial_a \phi \partial^a \phi - Q \hat{R} \phi + O(\Psi) f(\phi) \right) , \quad (2.2)$$

where

$$Q = \sqrt{\frac{25-c}{3}}$$

and $O(\Psi)$ is a matter primary field of dimension h . Without extra fine-tuning, the Liouville potential couples to the lowest dimension primary $O_{\min}(\Psi)$. It is not hard to work with a general primary field O , although occasionally we will specify our formulae to O_{\min} . The gravitational dressing function $f(\phi)$ is expected to have the large ϕ asymptotic form

$$f(\phi) \rightarrow A e^{\alpha_+ \phi} + B e^{\alpha_- \phi} \quad (2.3)$$

where

$$\alpha_\pm = -\frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} - 2 + 2h} \quad (2.4)$$

are the two solutions of the equation

$$2h - \alpha(\alpha + Q) = 2 .$$

This equation guarantees the conformal invariance of Liouville theory in the weakly

interacting region of large ϕ . If $A > 0$ then, as ϕ increases, the first term in eq. (2.3) rapidly becomes dominant. Furthermore, large values of ϕ are important in the path integral because they are not suppressed by the Liouville potential. Therefore, in the generic case $A > 0$, we may approximate

$$f(\phi) \sim \Delta e^{\alpha_+ \phi} .$$

Applying now the analysis of ref. [5] we find that the sum over surfaces of genus G obeys the scaling law

$$\frac{\partial^2 F_G}{\partial \Delta^2} \sim \frac{1}{\Delta^{2G+\gamma(1-G)}}$$

where the string susceptibility exponent is

$$\gamma = 2 + \frac{Q}{\alpha_+} . \quad (2.5)$$

Let us specify now to the case $O = O_{\min}$. Using eq. (1.4) and the formula for the central charge,

$$c = 1 - 6 \frac{(p-q)^2}{pq} ,$$

we obtain

$$\begin{aligned} Q &= \sqrt{2} \frac{p+q}{\sqrt{pq}} , \\ \alpha_+ &= -\frac{p+q-1}{\sqrt{2pq}} . \end{aligned} \quad (2.6)$$

Therefore, the string susceptibility exponent is

$$\gamma = -\frac{2}{p+q-1} \quad (2.7)$$

which agrees with the conventional matrix model results [12].

Our main observation is that, by fine-tuning the theory, we should be able to reach a phase where the dressing function is given by eq. (2.3) with $A = 0$, *i.e.*

$$f(\phi) \sim \Delta e^{\alpha_- \phi} . \quad (2.8)$$

We believe that, in the language of the matrix models, this fine-tuning is achieved by setting g , the coupling constant for touching interactions, to g_t . Indeed, the touching

interactions add tiny wormholes to surface geometry and, therefore, modify the ultraviolet (large ϕ) structure of the theory. It is reasonable that, by fine-tuning the ultraviolet boundary conditions, we may find a solution for the gravitational dressing with $A = 0$. While the precise mechanism for this is not entirely clear, we will simply check that the new critical behavior in the modified matrix models corresponds to the unconventional gravitational dressing, eq. (2.8).

The calculation of string susceptibility proceeds analogously to ref. [5], and we find a modified string susceptibility exponent

$$\gamma_{\text{modified}} = 2 + \frac{Q}{\alpha_-} . \quad (2.9)$$

Using eqs. (2.4) and (2.5), it is easy to establish that γ_{modified} and γ are related by

$$\frac{1}{\gamma_{\text{modified}} - 2} + \frac{1}{\gamma - 2} = -1 ,$$

which implies

$$\gamma_{\text{modified}} = \frac{\gamma}{\gamma - 1} . \quad (2.10)$$

This is a completely general relation, independent of which operator O enters the Liouville potential. One easily sees that, if γ is negative, then γ_{modified} is positive. Thus, positive γ_{modified} arise naturally in Liouville gravity! It is interesting that eq. (2.10) was recently obtained by Durhuus [13] on the basis of certain assumptions about random surfaces coupled to spin systems. Using purely combinatorial arguments he argued that, given a theory with scaling exponent γ , one should be able to construct a theory with scaling exponent $\frac{\gamma}{\gamma-1}$. We have shown how the modified scaling behavior, eq. (2.10), arises in Liouville gravity. We have also proposed a connection between the fine-tuning in modified matrix models and in Liouville gravity.

Specifying the theory to the case $O = O_{\text{min}}$, we find

$$\alpha_- = -\frac{p+q+1}{\sqrt{2pq}} . \quad (2.11)$$

Eq. (2.9) now gives

$$\gamma_{\text{modified}} = \frac{2}{p+q+1} . \quad (2.12)$$

Let us compare this with the known results from the modified matrix models. Consider,

for instance, the k -th multicritical one-matrix model which corresponds to the $(2, 2k-1)$ minimal model coupled to gravity. Without the touching interactions, $\gamma = -1/k$ in agreement with eq. (2.7). Eq. (2.12) predicts that, after a fine-tuning of the touching interactions, γ should jump to $1/(k+1)$. This is precisely what happens [7,8,9]. In our opinion, this provides serious evidence in favor of our interpretation of Liouville theory.

A similar argument applies to the $c = 1$ model, which is special because $\alpha_+ = \alpha_- = \sqrt{2}$. As a result, the Liouville interaction has the form [14]

$$\int d^2\sigma T(\phi) ,$$

$$T(\phi) \rightarrow A\phi e^{-\sqrt{2}\phi} + B e^{-\sqrt{2}\phi} .$$

Without fine-tuning, $A > 0$, and the first term dominates for large ϕ giving rise to scaling violations [14]. If, however, we reach a phase with $A = 0$, then the usual DDK analysis applies, and we find $\gamma = 0$ with no scaling violations [11]. This is precisely the new scaling behavior found in the $c = 1$ matrix model modified by the touching interactions [10,11].

The Liouville approach is valuable not only in providing a string theoretic framework for the matrix model results. Some calculations can be performed very efficiently starting directly from the Liouville path integral. Perhaps the simplest such calculations is the sum over surfaces of genus 1. In ref. [15,16] this path integral was calculated exactly, with the result

$$F_1 = \frac{(p-1)(q-1)}{24\sqrt{2pq}|\alpha|} |\log \Delta| . \quad (2.13)$$

For the conventional (p, q) models coupled to gravity we use $\alpha = \alpha_+$, eq. (2.6), and arrive at

$$F_1 = \frac{(p-1)(q-1)}{24(p+q-1)} |\log \Delta| , \quad (2.14)$$

which agrees with various matrix model results [17].^{*}

^{*} Results from matrix models with even potentials need to be divided by 2 to eliminate overcounting.

For the (p, q) models fine-tuned in the sense described above we may try to use $\alpha = \alpha_-$, eq. (2.11), in eq. (2.13). Then we find a contribution to the genus 1 free energy,

$$F_1^{\text{modified}} = \frac{(p-1)(q-1)}{24(p+q+1)} |\log \Delta| . \quad (2.15)$$

In ref. [8] the scaling of F_1^{modified} was studied in multicritical one-matrix models and was found to be $\sim |\log \Delta|$, in agreement with eq. (2.15). However, the dependence of the normalization on p and q has not been calculated in the matrix models. Unfortunately, eq. (2.15) is not the complete prediction of Liouville theory. Since the α_- dressed operators are “macroscopic”, with wave functions peaked in the strong coupling region [18], they are sensitive to non-trivial string loop corrections. Such corrections, which probably alter the coefficient of $|\log \Delta|$ in eq. (2.15), require a separate investigation.

3. Modified two-matrix models

In the previous section we proposed a Liouville gravity formulation of the matrix models with fine-tuned touching interactions and obtained a number of non-trivial predictions. In this section we study such modified matrix models for the $(p, p+1)$ conformal minimal models coupled to gravity. We calculate the sum over genus zero surfaces, $F_0(\Delta)$, and, after the fine-tuning, find $\gamma_{\text{modified}} = \frac{1}{p+1}$, in agreement with eq. (2.12).

As a warm-up, we repeat the calculation for pure gravity ($p = 2$), which may be described by the one-matrix model

$$Z = \int \mathcal{D}\Phi e^{-N \left[\text{Tr}(\Phi^2 - \lambda \Phi^4) - \frac{g}{2N} (\text{Tr} \Phi^4)^2 \right]} . \quad (3.1)$$

Comparing with ref. [7], we have replaced $(\text{Tr} \Phi^2)^2$ by $(\text{Tr} \Phi^4)^2$. This makes the calculation a bit simpler but, as expected, results in the same universal behavior. We will use a self-consistent method, analogous to Hartree-Fock calculations, to analyze eq. (3.1). For the purpose of finding the free energy to the leading order in N , it is permissible to

make a replacement [11]

$$\begin{aligned} (\text{Tr } \Phi^4)^2 &\rightarrow 2Nc \text{Tr } \Phi^4 - Nc^2 , \\ c &= \left\langle \frac{1}{N} \text{Tr } \Phi^4 \right\rangle . \end{aligned} \quad (3.2)$$

Substituting this into (3.1), we arrive at an auxiliary one-matrix model with coupling constant $\kappa = \lambda + gc$ and no touching interactions. Thus, the self-consistency condition has the form

$$c = \left\langle \frac{1}{N} \text{Tr } \Phi^4 \right\rangle (\kappa) . \quad (3.3)$$

The right-hand side is simply the puncture one-point function in the conventional one-matrix model with a $\kappa \text{Tr } \Phi^4$ interaction. Using well-known results, we find that eq. (3.3) becomes

$$c = a_1 - a_2(\kappa_c - \kappa) + a_3(\kappa_c - \kappa)^{3/2} + \dots$$

where a_i and κ_c are positive constants. Differentiating this with respect to λ , we find

$$\frac{\partial c}{\partial \lambda} (1 - a_2 g + \frac{3}{2} g a_3 (\kappa_c - \kappa)^{1/2}) = a_2 - \frac{3}{2} a_3 (\kappa_c - \kappa)^{1/2}$$

(from here on we retain only the leading singular terms). For $g < 1/a_2$, $\frac{\partial c}{\partial \lambda}$ is finite at the critical point $\kappa = \kappa_c$, and the scaling with $\gamma = -1/2$ follows. If, however, we fine-tune $g = 1/a_2$, then

$$\frac{\partial c}{\partial \lambda} = \frac{2}{3} \frac{a_2^2}{a_3 (\kappa_c - \kappa)^{1/2}} = \frac{2}{3} \frac{a_2^{5/2}}{a_3 (a_1 - c)^{1/2}} .$$

Setting

$$\lambda_c = \kappa_c - \frac{a_1}{a_2} , \quad \Delta = \lambda_c - \lambda ,$$

we have

$$\frac{\partial}{\partial \Delta} (a_1 - c)^{3/2} = \frac{a_2^{5/2}}{a_3} .$$

Therefore,

$$a_1 - c = \Delta^{2/3} \frac{a_2^{5/3}}{a_3^{2/3}}$$

Since $c = \frac{\partial F_0}{\partial \lambda}$ we finally obtain

$$\frac{\partial^2 F_0}{\partial \Delta^2} = -\frac{\partial c}{\partial \Delta} = \frac{2}{3} \frac{a_2^{5/3}}{a_3^{2/3}} \Delta^{-1/3}$$

which indicates that $\gamma_{\text{modified}} = 1/3$, in agreement with ref. [7].

Now let us proceed to the two-matrix model which describes an Ising spin ($p = 3$) coupled to gravity,

$$\begin{aligned} Z &= \int \mathcal{D}\Phi_1 \mathcal{D}\Phi_2 e^{-N [S(\Phi_1) + S(\Phi_2) + k \text{Tr} \Phi_1 \Phi_2]} , \\ S(\Phi) &= \text{Tr} (\Phi^2 - \lambda \Phi^4) - \frac{g}{2N} (\text{Tr} \Phi^4)^2 . \end{aligned} \quad (3.4)$$

If we set

$$c = \left\langle \frac{1}{N} \text{Tr} \Phi_1^4 \right\rangle = \left\langle \frac{1}{N} \text{Tr} \Phi_2^4 \right\rangle \quad (3.5)$$

and make the substitution (3.2) in eq. (3.4), then we arrive at an auxiliary two-matrix model with no touching interactions and coupling constant $\kappa = \lambda + gc$. From the fact that, for a specially chosen k , such a model has $\gamma = -1/3$ it follows that the self-consistency condition has the form

$$c = a_1 - a_2(\kappa_c - \kappa) + a_3(\kappa_c - \kappa)^{4/3} + \dots$$

where a_i and κ_c are a new set of positive constants. If we fine tune $g = 1/a_2$, then

$$\frac{\partial c}{\partial \lambda} = \frac{3}{4} \frac{a_2^2}{a_3(\kappa_c - \kappa)^{1/3}} = \frac{3}{4} \frac{a_2^{7/3}}{a_3(a_1 - c)^{1/3}} .$$

Integrating this equation, we find

$$a_1 - c = \Delta^{3/4} \frac{a_2^{7/4}}{a_3^{3/4}} .$$

Since $\frac{\partial F_0}{\partial \lambda} = 2c$, we finally obtain

$$\frac{\partial^2 F_0}{\partial \Delta^2} = \frac{3}{2} \Delta^{-1/4} \frac{a_2^{7/4}}{a_3^{3/4}}$$

which shows that $\gamma_{\text{modified}} = 1/4$, in agreement with eq. (2.12).

It is now clear how to generalize our methods to an arbitrary two-matrix model describing the $(p, p+1)$ minimal model coupled to gravity [19],

$$\begin{aligned} Z &= \int \mathcal{D}\Phi_1 \mathcal{D}\Phi_2 e^{-N [S_p(\Phi_1) + S_p(\Phi_2) + k \text{Tr} \Phi_1 \Phi_2]} , \\ S_p(\Phi) &= \text{Tr} (\Phi^2 - \lambda \Phi^4 + \dots + \tau \Phi^{2p-2}) - \frac{g}{2N} (\text{Tr} \Phi^4)^2 . \end{aligned} \quad (3.6)$$

The parameters of the potential $S_p(\Phi)$ can be tuned [19] in such a way that, for $g = 0$, $\gamma = -1/p$. For $g > 0$, the effective quartic coupling is $\kappa = \lambda + gc$, where c is defined by eq. (3.5). The self-consistency condition has the form

$$c = a_1 - a_2(\kappa_c - \kappa) + a_3(\kappa_c - \kappa)^{(p+1)/p} + \dots$$

If we fine-tune $g = 1/a_2$ and repeat the familiar steps, we arrive at

$$\frac{\partial^2 F_0}{\partial \Delta^2} \sim \Delta^{-1/(p+1)}$$

which implies $\gamma_{\text{modified}} = \frac{1}{p+1}$, once again in agreement with eq. (2.12).

The calculations in this section provide a check, in the context of unitary minimal models coupled to gravity, of the assertion that fine-tuning the touching interactions makes the susceptibility exponent jump from γ to $\frac{\gamma}{\gamma-1}$. Ref. [13] and the Liouville gravity arguments of sec. 2 strongly suggest that this phenomenon is completely general.

4. Discussion

In this paper we have proposed a Liouville gravity formulation of the matrix models with fine-tuned $g(\text{Tr } \Phi^n)^2$ terms. It involves theories with $c \leq 1$ characterized by the unconventional branch of gravitational dressing in the Liouville potential. Thus, the hope that these theories correspond to $c > 1$ is not realized. Nevertheless, our proposal opens many directions for future research which may shed new light on both matrix models and Liouville theory. First, it would be interesting to see how our arguments square with those of ref. [18]. Second, the calculation of string susceptibility is only a first step in comparing the discretized and continuum approaches. In fact, we should be able to extend most (if not all) the calculations in conventional matrix models to the modified matrix models. The obvious questions are correlation functions of scaling operators, higher-genus corrections, etc. Although some results are available [8,9], much progress remains to be made. It is clearly of interest to carry out parallel calculations in Liouville theory.

As remarked in sec. 2, some necessary Liouville gravity calculations are extensions of known results, *e.g.* the path integral at genus 1. Furthermore, the spectrum of operators can be read off from this path integral [16]. Every operator can be written as $\mathcal{O}e^{\alpha\phi}$ where \mathcal{O} depends on the matter, ghosts and the non-zero modes of ϕ . Using the techniques of ref. [16], one finds that the spectrum of dimensions of \mathcal{O} is the same as in the conventional Liouville theory. The only new feature is that the operator appearing in the action receives the α_- dressing. We believe that, without additional fine-tuning, all the other operators receive α_+ dressing. However, we may attempt to change the branch of dressing of other operators by further fine-tuning. It would be of interest to look for such effects in the matrix models.

A good motivation for studying operators with α_- dressing is that they arise in the black hole model [20,21], where the interaction term is essentially $\partial X \bar{\partial} X e^{-\sqrt{2}\phi}$. It would be very interesting to formulate a matrix model which describes such a two-dimensional black hole. There has been a number of proposals for such a matrix model [22,23], but it seems that fine-tuning the $g(\text{Tr } \Phi^n)^2$ terms offers some intriguing new possibilities.

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